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Geometrical Construction of Two-Dimensional and Axisymmetrical Flow Fields

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It is assumed that a potential flow field, consisting of lines of constant velocity potential and of streamlines, is approximated by a net of circular arcs that intersect at right angles. In general, this is possible with considerably greater accuracy than by using a net of straight elements with the same number of meshes. The method of constructing such a field, as derived in the following, is based on a theorem of elementary geometry which seems to have been unknown or at least has not been used in the field of fluid mechanics. It states that the corners of a rectangle formed by circular arcs are situated on a circle. A proof is given in this paper by conformal mapping. This method can be applied not only to two-dimensional but also to axisymmetrical flow of incompressible or compressible media. Its simplicity and the small overall number of steps generally required permit the solution of problems with boundary conditions where iteration is necessary. The paper discusses various examples and demonstrates the accuracy of the method in cases where an exact solution is known.

Nomenclature

ϕ	= velocity potential
ψ	= stream function
ds	= arc element of a streamline
dn	= arc element of an equal potential line
w	= velocity of flow
θ	= angle of flow direction with a fixed direction
r	= distance from axis of symmetry
ρ	= mass density
p	= pressure
T	= absolute temperature
c	= velocity of sound
M	= Mach number
$\gamma = c_p/c_v$	= ratio of specific heats

Subscripts denote special values, e.g., c_0 , ρ_0 , p_0 are the values of c , ρ , p for the gas at rest ($w = 0$).

Introduction

GEOMETRICAL methods of a stepwise character have not been applied very frequently in fluid mechanics and usually have been considered as laborious and rather inaccurate (see Ref. 1, p. 133), except perhaps the method of characteristics as used in gasdynamics for the solution of supersonic problems (refer to the list of textbooks on gasdynamics given in Ref. 2). On the other hand, the analytical methods also are usually laborious and, up to the present time, have not provided the solutions to a number of very interesting and important problems.

The lack of accuracy of stepwise geometrical flow construction methods usually stems from the approximation of equipotential lines and streamlines by piecewise straight elements. In order to increase the accuracy, it is necessary to increase the number of steps, in which case both the number of errors and the laboriousness of the procedure are increased. Therefore, it seemed to be appropriate to develop a method where the equipotential lines and the streamlines are approximated piecewise by circular arcs in such a way that the meshes of the net of these lines are replaced by rectangles

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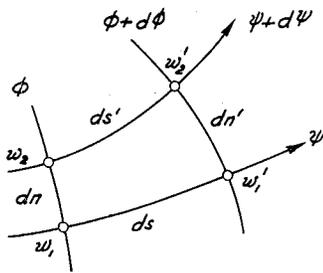


Fig. 1 Natural coordinates.

consisting of circular arcs. In general, such approximation is possible with considerably greater accuracy than by using a net of straight elements with the same number of meshes. The method of constructing a flow field as derived in the following is based on a theorem of elementary geometry which will be proved by conformal mapping. It states that *the corners of a rectangle formed by circular arcs are situated on a circle.*

The method is applicable in the two-dimensional and axis-symmetrical case of a flow of an incompressible, as well as a compressible, medium. Various applications such as flow through ducts and nozzles and around bodies are discussed briefly in this paper. The accuracy is checked in cases with analytically known exact solutions. The method can be combined with other methods. Thus, free-hand drawings of flow patterns can be corrected by such combination (compare Ref. 3). The present paper is an abridged presentation of an unpublished earlier investigation by the author.⁴

1. Two-Dimensional and Axisymmetrical Natural Equations of a Potential Flow

A steady two-dimensional potential flow of an incompressible medium is considered first. Two neighboring streamlines are denoted by ψ and $\psi + d\psi$. The distance between these, measured along an orthogonal trajectory to the streamlines, is denoted by dn (see Fig. 1). If on dn the average velocity of the flow is w , the flux through dn (the volume passing dn at unit time) is given by the value $w dn$, assuming that the flow occurs between two parallel planes with unit distance. Between the two streamlines ψ and $\psi + d\psi$, the flux through a cross section dn is the same at any place; therefore, the value $d\psi$ is determined by the well-known equation of continuity

$$d\psi = w dn \tag{1}$$

ϕ -values are assigned to the orthogonal trajectories of the streamlines so that ϕ and $\phi + d\phi$ denote two neighboring trajectories. They form with the streamlines ψ and $\psi + d\psi$ a rectangle of generally curved arcs that can be approximated by circular arcs. The arc length along the streamline between the trajectories ϕ and $\phi + d\phi$ is denoted by ds . The circulation of the flow around a point P is defined in the following way. An arbitrary simple curve C , for example, a circle around P as center, encloses the point P . The arc length of an element of C is denoted by $d\sigma$. The velocity vector w at a point on C is projected in the direction of $d\sigma$ and the length of the projection denoted by v . Then, the circulation of the flow around point P is defined by the sum

$$\oint_C v d\sigma = \Gamma \tag{2}$$

The flow is called irrotational if $\Gamma = 0$ for every curve C enclosing P . If C is the quadrangle formed by the arcs ϕ , $\phi + d\phi$, ψ , $\psi + d\psi$, the condition for an irrotational flow is that $w ds$ is constant along the streamlines between the trajectories ϕ and $\phi + d\phi$. If, therefore, $d\phi$ is identified by $w ds$, the condition for irrotational flow becomes

$$d\phi = w ds \tag{3}$$

Equations (1) and (3) are the natural equations of the two-

dimensional steady potential flow of an incompressible medium. They depend only on natural values and not on any arbitrarily chosen coordinate system. The ϕ - and ψ -values are uniquely determined except for arbitrary additive constants.

In the same way, the natural equations in the case of a steady two-dimensional potential flow of a compressible medium are obtained in the form

$$d\phi = w ds \quad d\psi = \rho w dn \tag{4}$$

where ρ is the (variable) mass density of the fluid. In the case of isentropic flow of a gas, these two-dimensional natural equations of gasdynamics can be written in the form

$$d\left(\frac{\phi}{c_0}\right) = \frac{w}{c_0} ds \quad d\left(\frac{\psi}{\rho_0 c_0}\right) = \frac{\rho}{\rho_0} \frac{w}{c_0} dn \tag{5}$$

where

$$\frac{\rho}{\rho_0} = \left[1 - \frac{\gamma - 1}{2} \left(\frac{w}{c_0}\right)^2\right]^{1/(\gamma - 1)} \tag{6}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma \tag{7}$$

$$\frac{T}{T_0} = \left(\frac{c}{c_0}\right)^2 = \left(\frac{\rho}{\rho_0}\right)^{\gamma - 1} \tag{8}$$

$$c_0 = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2} \quad \gamma = \frac{c_p}{c_v} \tag{9}$$

are well-known relations of thermodynamics for the mass density ρ , the temperature T , the pressure p , and the velocity c of sound of an isentropic gas. The subscript zero denotes the values of these variables for the gas at rest. The velocity ratio w/c_0 is related to the Mach number M by the relation

$$\frac{T}{T_0} = \frac{1}{1 + [(\gamma - 1)/2]M^2} \tag{10}$$

and Eqs. (8) and (6).

In the case of a steady axially symmetric potential flow of an incompressible frictionless medium, the distance r of a point from the axis of symmetry enters the natural equations of the flow which take the form

$$d\phi = w ds \quad d\psi = r w dn \tag{11}$$

where ψ is Stokes' stream function.

Finally, in the case of a steady axially symmetric gas flow, the natural equations are

$$d\left(\frac{\phi}{c_0}\right) = \frac{w}{c_0} ds \quad d\left(\frac{\psi}{r_0 \rho_0 c_0}\right) = \frac{\rho}{\rho_0} \frac{r}{r_0} \frac{w}{c_0} dn \tag{12}$$

2. Some Theorems on the Geometry of Circles

The streamlines $\psi = \text{const}$ and the lines of equal velocity potential $\phi = \text{const}$ form an orthogonal net of curves. Any sufficiently small mesh of this net can be considered as

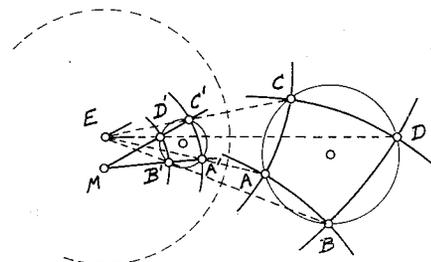


Fig. 2 Rectangles formed by circular arcs.

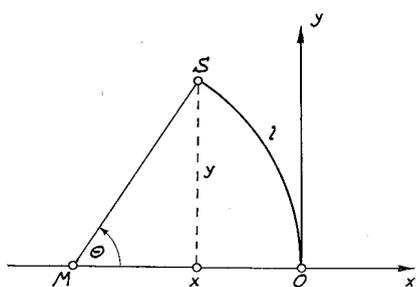


Fig. 3 Definition of the Cochleoid.

approximately formed by four pairwise perpendicular circular arcs. In general, such an approximation is very accurate and permits representation of a flow pattern by a rather small number of meshes consisting of rectangles of circular arcs. For any such mesh, theorem I holds.

Theorem I: The four corner points of a rectangle formed by circular arcs are situated on a circle.

In order to prove this, an intersection point of two opposite sides of the rectangle $ABCD$ (see Fig. 2), for example, point E , is transformed by inversion⁵ into the infinity. By this transformation, the two circles AB and CD are mapped into straight lines. Because the transformation leaves angles invariant, the circles AC and BD are mapped into two circles that are orthogonal to the two straight lines. Their center, therefore, is the point M of intersection of the two straight lines. The four points A', B', C', D' , which correspond to the corners A, B, C, D of the rectangle, are situated on a circle because of the symmetry of the configuration. Therefore, the points A, B, C, D also are situated on a circle that is the image of the circle through $A'B'C'D'$ due to the inversion.

Another theorem that will prove to be useful can be derived from a curve known under the name of Cochleoid. In order to define this curve, a circular arc of length l is considered to be perpendicular to a straight line at one of its endpoints O (Fig. 3). The locus of the other endpoint S is the Cochleoid that was discovered by Bernoulli. If the straight line is chosen as the x axis of a rectangular x, y coordinate system with the origin O , and if θ is the angle between the two radii determining the circular arc OS , then the coordinates of S as functions of the parameter θ are

$$x = (l/\theta)(\cos\theta - 1) \quad y = (l/\theta) \sin\theta \quad (13)$$

Figure 4 shows the curve and its osculating circle at its highest point $y = l$. The latter one can be obtained by expanding

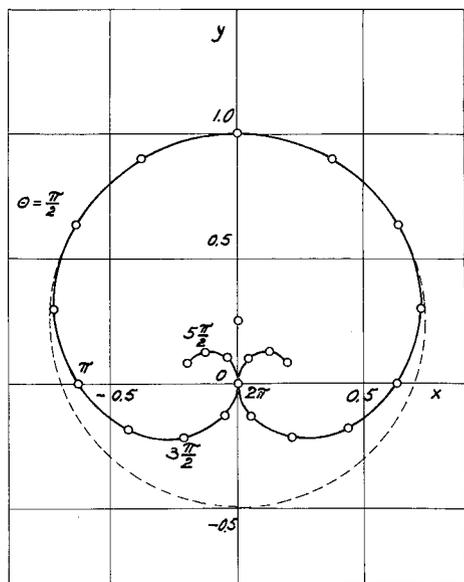


Fig. 4 Cochleoid.

$\cos\theta$ and $\sin\theta$ in series of powers of θ . For sufficiently small θ , one obtains

$$x = l(\theta/2) \quad y = l[1 - (\theta^2/6)] \quad (14)$$

The curvature $1/R$ follows from

$$\frac{1}{R} = -\frac{4}{3} \frac{1}{(1 + \frac{4}{3}\theta^2)^{3/2}} \quad (15)$$

It is $\theta = 0$ for $y = l$ and, therefore, the radius of curvature

$$R_0 = -\frac{3}{4}l \quad (16)$$

The center of the osculating circle has the coordinate

$$y_0 = l/4 \quad (17)$$

Thus, theorem II has been obtained.

Theorem II: In order to plot a given length $l = s$ along the circumference of a given circle starting from a point O , construct the tangent at O and along the tangent the length $OQ = s/4$. The circle with the center Q and the radius $\frac{3}{4}s$ intersects the given circle at a point P , which is approximately the distance s from O , measured along the circle (see Fig. 5).

The approximations (14) for Eq. (13) have been derived by neglecting fourth and higher powers of θ . Therefore, the accuracy of the construction can be expected to be an excellent one. In Fig. 4, the dashed osculating circle actually coincides with the curve up to θ -values of about 90° .

This construction can be used in order to draw a curve whose natural equation $1/R = f(s)$ is given, R being the radius of curvature at a point of the curve corresponding to the arc length s . Figure 5 actually represents the first step of this construction. The next step would consist of the analog construction starting with the tangent at P instead of O .

Theorem II also can be used in the opposite manner for determining the arc length OP of a circle by geometrical construction (Fig. 6). For this purpose, on the secant OP the points D_1 and D_2 must be constructed so that $OD_1 : D_1P = OD_2 : D_2P = 1 : 3$. Then D_1 and D_2 are the points that divide the stretch OP inside and outside in the ratio 1:3. The circle with the diameter $D_1 D_2$ intersects the tangent of point O of

Fig. 5 Approximate construction of a circular arc with given length.

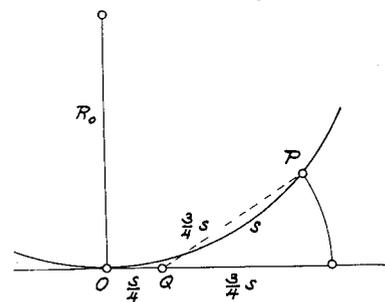
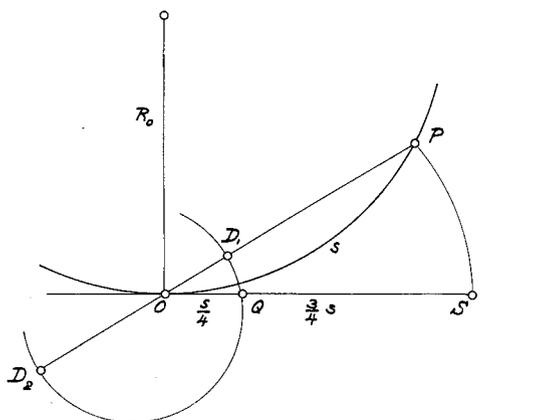


Fig. 6 Approximate determination of arc length.



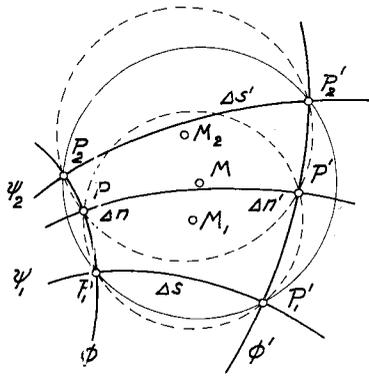


Fig. 7 Notations.

the given circle at the point Q for which $OQ = s/4$. The circle with the center Q and the radius OP finally intersects the tangent at a point S that has the distance s from O .

3. Solution of Basic Flow Problems by Stepwise Construction

The geometrical results derived in the preceding section now will be applied to the stepwise construction of the solution of flow problems.

Problem 1: Two streamlines $\psi = \psi_1 = \text{const}$ and $\psi = \psi_2 = \text{const}$ and an orthogonal trajectory $\phi = \text{const}$ are given. The flow velocities w_1 and w_2 at the intersections of $\phi = \text{const}$ with the two streamlines are also given. The flow velocities along the two streamlines are to be determined.

First, the case is considered of the two-dimensional flow of an incompressible medium for which the basic equations (3) and (1) hold. An arbitrary circle (see Fig. 7) passing through the intersections of the streamlines $\psi_1 = \text{const}$ and $\psi_2 = \text{const}$ with the orthogonal trajectory $\phi = \text{const}$ (chosen as equipotential line) has two other intersections through which an equipotential line $\phi' = \text{const}$ passes according to the geometrical theorem I. The unknown velocities in the latter intersections are denoted by w_1' and w_2' , respectively. The lengths of the sides of the mesh determined by the four intersections are denoted by Δs , Δn , $\Delta s'$, and $\Delta n'$. The average velocity† on Δs equals $(w_1 + w_1')/2$, on Δn equals $(w_1 + w_2)/2$, on $\Delta s'$ equals $(w_2 + w_2')/2$, and on $\Delta n'$ equals $(w_1' + w_2')/2$. The basic equations, written as difference equations, therefore yield

$$\Delta\phi = \frac{w_1 + w_1'}{2} \Delta s = \frac{w_2 + w_2'}{2} \Delta s' \quad (18)$$

$$\Delta\psi = \frac{w_1 + w_2}{2} \Delta n = \frac{w_1' + w_2'}{2} \Delta n' \quad (19)$$

The arc lengths Δs , Δn , $\Delta s'$, $\Delta n'$, considered as circular arcs, are known and can be found by construction according to the geometrical theorem II. Because the velocities w_1 and w_2 are also known, the unknown velocities w_1' and w_2' can be computed from (18) and (19). Thus,

$$w_1' = \frac{[w_2 + (w_1 + w_2)(\Delta n/\Delta n')]\Delta s' - w_1\Delta s}{\Delta s' + \Delta s} \quad (20)$$

$$w_2' = (w_1 + w_2)(\Delta n'/\Delta n) - w_1 \quad (21)$$

The arbitrary circle, used in this consideration, must be chosen so that the arcs on the streamline within this circle are approximately circular arcs. It has been assumed also that within this circle the arcs of the lines $\phi = \text{const}$ and $\phi' = \text{const}$ can be considered as circular arcs. Moreover, it has been assumed that the average velocities on these arcs can be considered as the arithmetic mean values of the velocities of

† The average velocity need not be assumed only as an arithmetical mean value. Any other mean value can also be used.

their endpoints with sufficient accuracy. These assumptions are satisfied if the streamlines ψ_1 and ψ_2 are neighboring streamlines, if the auxiliary circle intersects these streamlines at points near the points of intersection with the line $\phi = \text{const}$, and if the difference of the given velocities w_1 and w_2 is small.

The auxiliary circle yields only two points of the curve $\phi' = \text{const}$, namely, the points of intersection with the two streamlines. Another point can be constructed by choosing a point P on the line $\phi = \text{const}$ between P_1 and P_2 (see Fig. 7). Then, the circles through P_1PP_1' and P_2PP_2' intersect at a point P' situated on the equipotential line $\phi' = \text{const}$ and on the streamline through point P . In the same way more points of $\phi' = \text{const}$ and more streamlines can be constructed. The latter ones are the orthogonal circular arcs of ϕ and ϕ' passing through the points P .

In the cases of the two-dimensional potential flow of a compressible medium, the solution of the same problem can be obtained in a quite similar way. Instead of Eqs. (18) and (19), one has

$$\Delta\left(\frac{\phi}{c_0}\right) = \frac{w_1' + w_1}{2c_0} \Delta s = \frac{w_2 + w_2'}{2c_0} \Delta s' \quad (22)$$

$$\Delta\left(\frac{\psi}{\rho_0 c_0}\right) = \frac{\rho_{12} w_1 + w_2}{\rho_0} \Delta n = \frac{\rho_{12}' w_1' + w_2'}{2\rho_0} \Delta n' \quad (23)$$

where, according to (16), for an isentropic flow

$$\frac{\rho_{12}}{c_0} = \left[1 - \frac{\gamma - 1}{2} \left(\frac{w_1 + w_2}{2c_0}\right)^2\right]^{1/(\gamma-1)} \quad (24)$$

$$\frac{\rho_{12}'}{c_0} = \left[1 - \frac{\gamma - 1}{2} \left(\frac{w_1' + w_2'}{2c_0}\right)^2\right]^{1/(\gamma-1)} \quad (25)$$

Again Δs , Δn , $\Delta s'$, $\Delta n'$, and w_1 , w_2 are known. Using a graph for $(\rho/\rho_0)(w/c_0)$ as a function of w/c_0 , the expression $(\rho_{12}/\rho_0) \times [(w_1 + w_2)/2c_0]$ in Eq. (23) can be determined for the given $(w_1 + w_2)/2$. The same graph yields the value of $(w_1' + w_2')/2 = w'$ from the value of $(\rho_{12}'/\rho_0)[(w_1' + w_2')/2c_0]$ known from Eq. (23). Equation (22) then yields

$$w_1' = \frac{(w_2 + 2w')\Delta s' - w_1\Delta s}{\Delta s' + \Delta s} \quad (26)$$

and

$$w_2' = 2w' - w_1' \quad (27)$$

The problem can be solved in the same way for any density ρ that is a given function of the velocity w using a graph for ρw as function of w .

In the case of an axially symmetric potential flow of an incompressible medium, the corresponding formulas are

$$\Delta\phi = \frac{w_1 + w_1'}{2} \Delta s = \frac{w_2 + w_2'}{2} \Delta s' \quad (28)$$

$$\Delta\psi = \frac{r_1 + r_2}{2} \frac{w_1 + w_2}{2} \Delta n = \frac{r_1' + r_2'}{2} \frac{w_1' + w_2'}{2} \Delta n' \quad (29)$$

and, therefore,

$$w_1' = \frac{\{[(r_1 + r_2)/(r_1' + r_2')](\Delta n/\Delta n')(w_1 + w_2)\} \Delta s' - w_1\Delta s}{\Delta s' + \Delta s} \quad (30)$$

$$w_2' = \frac{r_1 + r_2}{r_1' + r_2'} \frac{\Delta n}{\Delta n'} (w_1 + w_2) - w_1' \quad (31)$$

Finally, in the axially symmetric case of compressible flow, it is

$$\Delta\left(\frac{\phi}{c_0}\right) = \frac{w_1 + w_1'}{2c_0} \Delta s = \frac{w_2 + w_2'}{2c_0} \Delta s' \quad (32)$$

$$\Delta\left(\frac{\psi}{\rho_0 c_0}\right) = \frac{r_1 + r_2}{2} \frac{\rho_{12}}{\rho_0} \frac{w_1 + w_2}{2c_0} \Delta n = \frac{r_1' + r_2'}{2} \frac{\rho_{12}'}{\rho_0} \frac{w_1' + w_2'}{2c_0} \Delta n' \quad (33)$$

From the graph for ρw as a function of w , Eq. (33) yields $(w_1' + w_2')/2 = w'$ in the same way as before. Then

$$w_1' = \frac{(w_2 + 2w')\Delta s' - w_1\Delta s}{\Delta s' + \Delta s} \quad (34)$$

and

$$w_2' = 2w' - w_1' \quad (35)$$

Problem 2: Two equal potential lines $\phi = \phi_1 = \text{const}$ and $\phi = \phi_2 = \text{const}$ and an orthogonal trajectory $\psi = \text{const}$ are given. Also given are the flow velocities w_1 and w_2 at the intersections of the streamline with the two equipotential lines. To be determined are the flow velocities along the lines $\phi_1 = \text{const}$ and $\phi_2 = \text{const}$.

This problem can be solved in the same way as problem 1, because the roles of the lines $\phi = \text{const}$ and $\psi = \text{const}$ are only exchanged. Somewhat different, however, is the solution of the next problem.

Problem 3: Given are a streamline and the velocity distribution along it. A point P_2 in a neighborhood of this streamline and the velocity at this point are also given. The streamline through point P_2 and the velocity along it have to be determined.

For the solution, the denotations of Fig. 7 can be used. Accordingly, $\psi = \psi_1 = \text{const}$ is assumed to be the streamline along which the velocity distribution is known. The point P_2 is in the neighborhood of the streamline with the known velocity w_2 . It is assumed that P_1P_1' is approximately a circular arc replacing the streamline between P_1 and P_1' . Furthermore, P_1 is assumed to be the endpoint of the circular arc through P_2 which is orthogonal to P_1P_1' . The arc lengths $P_1P_1' = \Delta s$ and $P_1P_2 = \Delta n$ and the velocities w_1, w_1' and w_2 of P_1P_1' and P_2 , respectively, are known. Unknown are the fourth corner point P_2' of the mesh $P_1P_1'P_2P_2'$ and the velocity at P_2' . The unknown corner point P_2' is situated on the circle through the points P_1P_1' and P_2 . The choice of a point P_2' on this circle determines the mesh $P_1P_1'P_2P_2'$. The lengths of the arcs $P_1P_1' = \Delta s$ and $P_1P_2 = \Delta n$ and the given velocities in the points P_1, P_1', P_2 determine the values $\Delta\phi$ and $\Delta\psi$ for the mesh $P_1P_1'P_2P_2'$. From these values and the arc lengths $\Delta s' = P_2P_2'$ and $\Delta n' = P_1'P_2'$, two conditions for the same velocity w_2' at P_2' are obtained. By interpolation, the point P_2' must be determined for which both conditions yield the same value. In the same way, the following problem can be solved by exchanging the roles of ϕ and ψ .

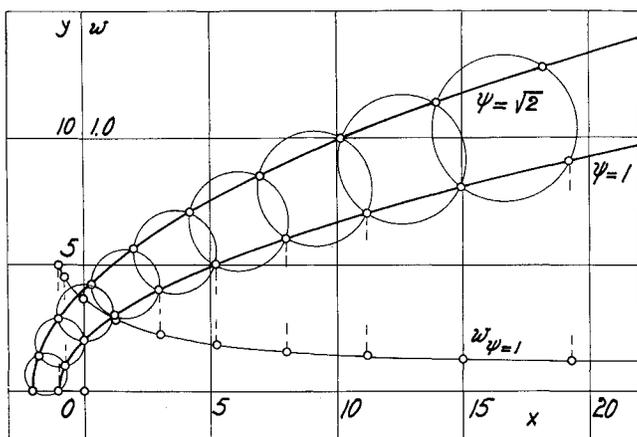


Fig. 8 Two-dimensional parabolic flow.

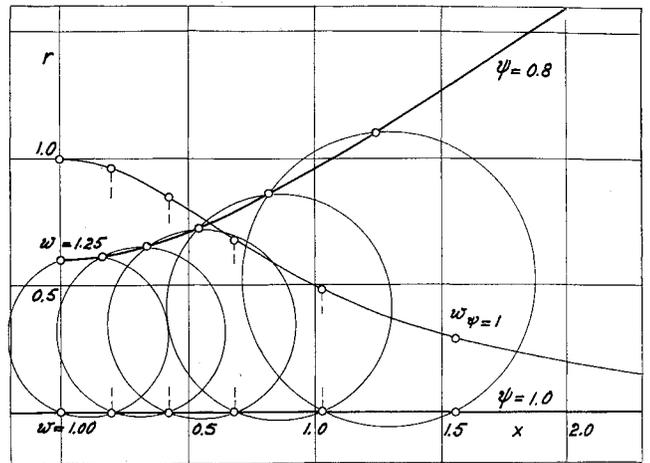


Fig. 9 Incompressible flow through axisymmetrical throat.

Problem 4: Given is an equipotential line and the velocity distribution along it. Given further are a point P_1' in a neighborhood of this line and the velocity at this point. The equipotential line through point P_1' and the velocity distribution along it are to be determined.

4. Applications

The following examples have been selected partly in order to show the accuracy that can be achieved with the method developed in the foregoing and partly in order to demonstrate approximate solutions of problems with unknown exact solutions. The method will be applied first to some problems for which exact analytical solutions exist.

1) The streamlines of the two-dimensional potential flow of an incompressible medium represented by

$$x = \phi^2 - \psi^2 \quad y = 2\phi\psi \quad (36)$$

are confocal parabolas. The velocity of the flow at any point x, y is

$$w = \frac{1}{2(\phi^2 + \psi^2)^{1/2}} \quad (37)$$

Figure 8 shows the streamlines $\psi = 1$ and $\psi = 2^{1/2}$. The velocities at the intersections with the equipotential line $\phi = 0$ are $w_1 = 0.500$ (on $\psi = 1$) and $w_2 = 0.354$ (on $\psi = 2^{1/2}$). Figure 8 also shows the velocity distribution along the streamline $\psi = 1$ computed from formula (37). The solution of problem 1 has been applied to this case choosing $\psi_1 = 1, \psi_2 = 2^{1/2}, w_1 = 0.5, w_2 = 0.354$, the given orthogonal trajectory of the two streamlines being the x axis $\phi = 0$. The circles shown in Fig. 8 are the auxiliary circles used for the stepwise construction of the velocity distribution along the streamline $\psi = 1$. The small circles on $\psi = 1$ indicate points at which the computed velocity distribution has been found by construction. The accuracy is a perfect one, although the number of steps used is small.

2) The axially symmetric potential flow through a circular hole in an infinite plane plate is given by

$$x = -\psi \operatorname{tg} \phi \quad r = \left(2 \frac{1 - \psi^2}{1 + \cos 2\phi}\right)^{1/2} \quad (38)$$

It has streamlines that are hyperbolas whose foci are situated on the circular edge of the hole. The velocity of the flow is determined by

$$w = \frac{\{\psi^2 + [(1 - \psi^2)/2](1 - \cos 2\phi)\}^{1/2}}{[2\psi^2/(1 + \cos 2\phi)] + [(1 - \psi^2)/2](1 - \cos 2\phi)} \quad (39)$$

Figure 9 shows the streamline $\psi = 0.8$ and the velocity dis-

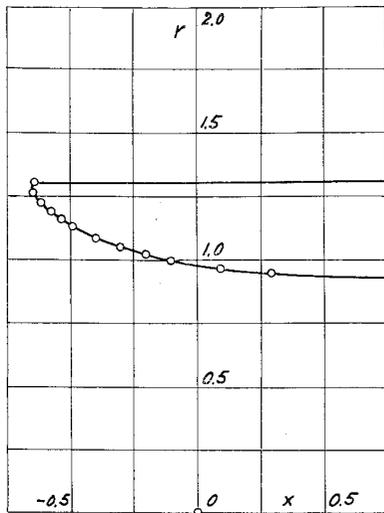


Fig. 10 Axisymmetrical Borda-mouthpiece.

tribution along the x axis which is a streamline ($\psi = 1$) and the axis of symmetry. The geometrical solution of problem 1 is applied to the axially symmetrical channel formed by rotation of the streamline $\psi = 0.8$ for $x \geq 0$ about the axis of symmetry. The inlet conditions on the r axis are $w = 1$ (for $\psi = 1$) and $w = 1.25$ (for $\psi = 0.8$). Again the points on $w_{\psi=1}$ are points of the velocity distribution along the x axis obtained by the geometrical method. The circles shown in Fig. 9 are the auxiliary circles used for the determination of the velocity. In spite of the width of the channel and the strong curvature of the streamline $\psi = 0.8$, the velocity points found by construction coincide very well with the velocity points obtained by exact analytical computation.

3) The two-dimensional potential flow entering the channel between two flat plates and forming between the plates a contour along which the fluid pressure has a constant value can be determined by conformal mapping using the hodograph method. The resulting shape is known as a Borda mouthpiece. Figure 10 shows the axially symmetric inlet with constant pressure distribution along the curved inlet walls obtained by stepwise application of the geometric method (compare with Ref. 13).

4) The accuracy of the result in these examples leaves no doubt that the method can be applied with similar accuracy also to problems of gasdynamics. Figure 11, for example, is an axially symmetric hyperbolic throat showing the dependence of the Mach number distribution along the axis and at the wall from the conditions at the entrance into the throat (compare with Refs. 8-12).[‡]

5. Generalizations

The geometrical method for the construction of two-dimensional or axisymmetrical flow fields as developed in the preceding sections can be generalized in various ways. The existence of a stream function ψ is a consequence of the continuity of the flow only. There exist generally orthogonal trajectories $\phi = \text{const}$ to the streamlines $\psi = \text{const}$. They form, with the lines $\psi = \text{const}$, meshes that can be approximated by meshes of rectangular circular arcs such as before. Their corners are situated on a circle according to theorem I. Besides the continuity equation, the method requires, in the case of incompressible flow, the knowledge of the circulation around each mesh (or vorticity) and, in the case of compressible flow, the additional knowledge of the fluid density ρ as a given function of the flow velocity w only. In the case of

[‡] The method also has been tested in the case of the transonic flow formerly published by the author (see Refs. 6 and 7). The accordance between the geometrical construction and the exact analytical solution proved to be an excellent one in spite of a limited number of steps.

potential flow the circulation was known to be zero, and for compressible flow the adiabatic relation between density and velocity was assumed.

Consider as an example of a generalization the case of a two-dimensional flow of an incompressible nonviscous medium. Then

$$(\partial^2\psi/\partial x^2) + (\partial^2\psi/\partial y^2) = f(\psi) \tag{40}$$

where f is an arbitrary function of ψ , and

$$\zeta = -f(\psi) \tag{41}$$

is the vorticity of the flow. For the circulation Γ , the relation

$$\Gamma = \oint_C udx + vdy = \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy = \iint_S \zeta dS \tag{42}$$

holds where the curve C encloses the area S . Thus, for a mesh

$$\Delta\Gamma = -f(\psi)\Delta S \tag{43}$$

A simple solution of Eq. (40) is

$$\psi = e^{ax} \sin by \tag{44}$$

for which

$$(\partial^2\psi/\partial x^2) + (\partial^2\psi/\partial y^2) = (a^2 - b^2)\psi \tag{45}$$

Thus,

$$\zeta = (b^2 - a^2)\psi \tag{46}$$

From

$$u = \partial\psi/\partial y \quad v = -\partial\psi/\partial x \tag{47}$$

follows, for the velocity,

$$w = e^{ax} [b^2 + (a^2 - b^2) \sin^2 by]^{1/2} \tag{48}$$

where $a = b$ yields a potential flow.

Figure 12 shows, for $a = 0.5$ and $b = 1$, the streamlines $\psi = 0$ and $\psi = 0.5$ forming a convergent duct in positive x direction with the corresponding velocity distributions. The velocity values obtained by application of the geometrical method follow from (see Fig. 1)

$$\Delta\psi = \frac{w_1 + w_2}{2} \Delta n = \frac{w_1' + w_2'}{2} \Delta n' \tag{49}$$

$$\Delta\Gamma = \frac{w_1 + w_1'}{2} \Delta s - \frac{w_2 + w_2'}{2} \Delta s' = (a^2 - b^2)\psi \tag{50}$$

where $a^2 - b^2 = -0.75$ and $\psi = 0.25$ is the mean value of

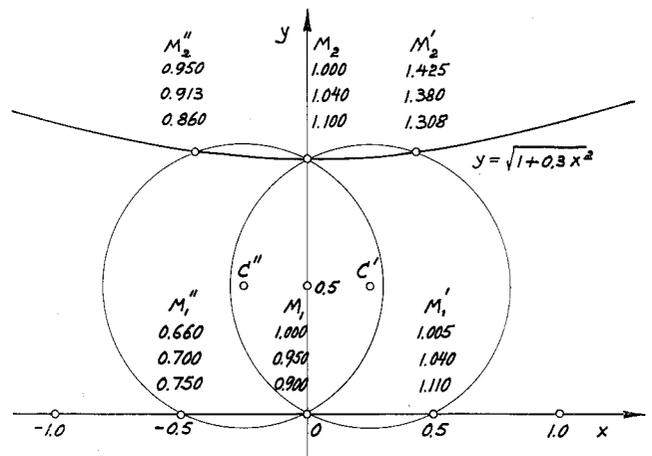


Fig. 11 Transonic flow.

ψ for the given streamlines $\psi = 0$ and $\psi = 0.5$. The inlet velocities w_1 and w_2 are prescribed values taken from the analytical solution. The area ΔS can be computed approximately from

$$\Delta S = \frac{\Delta n + \Delta n'}{2} \cdot \frac{\Delta s + \Delta s'}{2} \quad (51)$$

The resulting velocities w_1', w_2' are in complete agreement with the analytical solution.

Another generalization of this geometrical method concerns a two-phase flow. The construction of the flow in both phases can be carried out without further explanations. At the interphase, however, in the two-dimensional case, the boundary condition

$$p_1 - p_2 = T/R \quad (52)$$

must be satisfied, where $1/R$ is the curvature of the boundary, T is the surface tension, and p_1, p_2 are the pressures adjacent to the boundary. In the axisymmetrical case, formula (52) must be replaced by

$$p_1 - p_2 = T[(1/R_1) + (1/R_2)]$$

where $1/R_1$ and $1/R_2$ are the principle curvatures of the interphase.

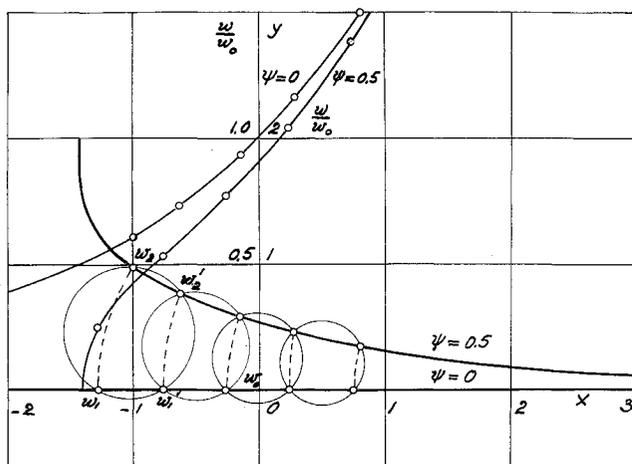


Fig. 12 Example of rotational flow.

Conclusion

The geometrical method, developed in this paper for the stepwise construction of two-dimensional or axially symmetric potential flows of incompressible or compressible media, based on natural coordinates and some results of the geometry of circles, has been applied to some problems with analytically known, as well as unknown, solutions. The displayed accuracy of the approximations and the relative simplicity of the constructions combined with the elementary nature of the computations suggest many other applications and extensions to other types of flow. Subsequent papers will deal with such applications and generalizations.

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